Exam Calculus 2

11 April 2023, 18:15-20:15

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points. Calculators, books and notes are not permitted.

1. [**6+6+8=20 Points**] Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{x^3+xy^2+2x^2+2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases},$$

where $c \in \mathbb{R}$.

   (a) Determine c such that f becomes continuous at $(x,y) = (0,0)$.

   (b) For the value of c found in part (a) and $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ a unit vector, determine the directional derivative $D_{\boldsymbol{u}} f(0,0)$.

   (c) Use the definition of differentiability to show that for the value of c found in part (a), the function f is differentiable at $(x,y) = (0,0)$ and determine the derivative of f at $(x,y) = (0,0)$.

2. [**5+7+8=20 Points**] Let $S \subset \mathbb{R}^2$ be the ellipse defined by the equation

$$S = \{(x,y) \in \mathbb{R}^2 \,|\, 3x^2 + 2xy + 3y^2 = 16\}.$$

   (a) For each point $(x_0, y_0) \in S$, determine the tangent line of S at (x_0, y_0).

   (b) Use the Implicit Function Theorem to determine the points (x_0, y_0) in S where S can be considered to be locally the graph of a function f of x. At such points, compute the derivative of f and show that the tangent line found in (a) coincides with the graph of the linearization of f.

   (c) Use the method of Lagrange multipliers to determine the points on the ellipse S that are closest to and furthest away from the origin, respectively.

3. [**8+12+5=25 Points**] For constants $a, b \in \mathbb{R}$, define the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\mathbf{F}(x,y,z) = ax\sin(\pi y)\,\mathbf{i} + (x^2\cos(\pi y) + bye^{-z})\,\mathbf{j} + y^2 e^{-z}\,\mathbf{k}$$

for $(x,y,z) \in \mathbb{R}^3$.

   (a) Show that $\mathbf{F}$ to be conservative requires $a = 2/\pi$ and $b = -2$.

   (b) Determine a scalar potential for $\mathbf{F}$ for the values of a and b given in part (a).

(c) For the values of a and b given in part (a), compute the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the curve parametrized by

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^2 t\,\mathbf{j} + \sin(2t)\,\mathbf{k}$$

with $t \in [0, \pi/2]$.

4. **[25 Points]** Let $S \subset \mathbb{R}^3$ be the quarter of the unit disk contained in the first quadrant of the (y, z)-plane, i.e. $S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0,\ y^2 + z^2 \leq 1 \text{ and } y, z \geq 0\}$. Let S be oriented by the unit vector $\mathbf{i}$. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ for $(x, y, z) \in \mathbb{R}^3$. Verify Stokes' Theorem for the given surface S and vector field $\mathbf{F}$ by computing both sides of the equality $\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.

Solutions

1. (a) In order to determine the limit of f at $(x, y) = (0, 0)$ we use polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ for $(x, y) \neq (0, 0)$. Then

$$\begin{aligned} f(x,y) &= \frac{r^3\cos^3\theta + r^3\cos\theta\sin^2\theta + 2r^2\cos^2\theta + 2r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= r(\cos^3\theta + \cos\theta\sin^2\theta) + 2. \end{aligned}$$

Considering the limit $r \to 0$ yields that f becomes continuous at $(x, y) = (0, 0)$.

(b) Let $\boldsymbol{u} = (v, w) \in \mathbb{R}^2$ with $v^2 + w^2 = 1$. Then

$$\begin{aligned} D_{\boldsymbol{u}}f(0,0) &= \lim_{h\to 0}\frac{f(hv,hw) - f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{h^3v^3 + h^3vw^2 + 2h^2v^2 + 2h^2w^2}{h^2(v^2+w^2)} - 2\right) \\ &= \lim_{h\to 0}\frac{1}{h^3}\left(h^3v^3 + h^3vw^2 + 2h^2 - 2h^2\right) \\ &= \lim_{h\to 0} v^3 + vw^2 \\ &= v^3 + vw^2 \\ &= v(v^2+w^2) \\ &= v, \end{aligned}$$

where in the third and last equality we used $v^2 + w^2 = 1$.

(c) According to part (b) we have $f_x(0,0) = 1$ (choose $\boldsymbol{u} = (v, w) = (1, 0)$) and $f_y(0,0) = 0$ (choose $\boldsymbol{u} = (v, w) = (0, 1)$). So the linearization of f at $(x, y) = (0, 0)$ is given by

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 2 + x.$$

For the differentiability of f at $(0, 0)$ we have that the limit of

$$\frac{f(x,y) - L(x,y)}{\|(x,y) - (0,0)\|}$$

is 0 for $(x, y) \to (0, 0)$. For $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} \frac{f(x,y) - L(x,y)}{\|(x,y) - (0,0)\|} &= \frac{1}{(x^2+y^2)^{1/2}}\left(\frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2} - (2+x)\right) \\ &= \frac{1}{(x^2+y^2)^{3/2}}\left(x^3 + xy^2 + 2x^2 + 2y^2 - 2(x^2+y^2) - x(x^2+y^2)\right) \\ &= \frac{1}{(x^2+y^2)^{3/2}}(0) \end{aligned}$$

which converges to 0 as $(x, y) \to (0, 0)$. The function f is hence differentiable at $(x, y) = (0, 0)$.

The derivative is

$$\nabla f(0,0) = (f_x(0,0), f_y(0,0)) = (1, 0).$$

2. (a) S is the level set of the function $F : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto F(x, y) = 3x^2 + 2xy + 3y^2$ for the value 16. For $(x_0, y_0) \in S$, $\nabla F(x_0, y_0)$ is perpendicular to S at (x_0, y_0). This gives the tangent line of S at (x_0, y_0)

$$F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) = 0.$$

Using $\nabla F(x_0, y_0) = (6x_0 + 2y_0, 6y_0 + 2x_0)$ we get the tangent line

$$\begin{aligned} & (6x_0 + 2y_0)(x - x_0) + (6y_0 + 2x_0)(y - y_0) = 0 \\ \Leftrightarrow\ & (6x_0 + 2y_0)x + (6y_0 + 2x_0)y = 6(x_0^2 + y_0^2) + 4x_0y_0. \end{aligned}$$

(b) For $(x_0, y_0) \in S$ to be a point where S is locally the graph of a function $x \mapsto f(x)$, (x_0, y_0) needs to satisfy

$$F_y(x_0, y_0) \neq 0$$

with F defined as in part (a). Hence at $(x_0, y_0) \in S$ with $2x_0 + 6y_0 \neq 0$, S is locally a graph over the x-axis.
Let $(x_0, y_0) \in S$ be such a point. Then

$$f'(x_0) = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)} = -\frac{6x_0 + 2y_0}{6y_0 + 2x_0}.$$

The linearization of f at x_0 is

$$L(x) = f(x_0) + f'(x_0)(x - x_0) = y_0 - \frac{6x_0 + 2y_0}{6y_0 + 2x_0}(x - x_0).$$

The graph of L is given by the equation

$$L(x) = y \Leftrightarrow y - y_0 = -\frac{6x_0 + 2y_0}{6y_0 + 2x_0}(x - x_0)$$

which after after multiplication by $6y_0 + 2x_0$ gives the equation for the tangent line found in part (a).

(c) We need to find the extrema of $(x, y) \mapsto g(x, y) = x^2 + y^2$ under the constraint $F(x, y) = 16$ with F defined as in part (a). By the Theorem on Lagrange Multipliers there exists a $\lambda \in \mathbb{R}$ such that $\nabla F(x, y) = \lambda \nabla g(x, y)$ at each such extremum . This gives the set of equations

$$\begin{array}{rcl} F_x(x,y) & = & \lambda g_x(x,y) \\ F_y(x,y) & = & \lambda g_y(x,y) \\ F(x,y) & = & 16 \end{array} \Leftrightarrow \begin{array}{rcl} 6x + 2y & = & 2\lambda x \\ 6y + 2x & = & 2\lambda y \\ 3x^2 + 2xy + 3y^2 & = & 16 \end{array}.$$

We can exclude $x = 0$ as this would also give $y = 0$ by the first equation which together do however not satisfy the third equation. Similarly we can exclude $y = 0$. For $(x, y) \neq (0, 0)$, we get from the first two equations

$$\lambda = \frac{6x + 2y}{2x} = \frac{6y + 2x}{2y}.$$

Hence

$$3 + \frac{y}{x} = 3 + \frac{x}{y} \Leftrightarrow \frac{y}{x} = \frac{x}{y} \Leftrightarrow x = \pm y.$$

The third equation then yields

$$\begin{aligned} & 3x^2 \pm 2x^2 + 3x^2 = 16 \\ \Leftrightarrow\ & 8x^2 = 16 \text{ for } x = y \text{ or } 4x^2 = 16 \text{ for } x = -y \\ \Leftrightarrow\ & x = \pm\sqrt{2} \text{ for } x = y \text{ or } x = \pm 2 \text{ for } x = -y. \end{aligned}$$

As $g(\pm\sqrt{2}, \pm\sqrt{2}) = 4$ and $g(\pm 2, \mp 2) = 8$ and g needs to attain its extrema on the compact set S by the WeierstraßExtreme Value Theorem, the points $(x, y) \in S$ closest to the origin are $(x, y) = \pm\sqrt{2}, \pm\sqrt{2})$ and the points furthest away from the origin are $(x, y) = (\pm 2, \mp 2)$.

3. (a) The curl of F is

$$\begin{aligned} \nabla \times \mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax\sin(\pi y) & x^2\cos(\pi y) + bye^{-z} & y^2e^{-z} \end{vmatrix} \\ &= (2ye^{-z} + bye^{-z})\,\mathbf{i} + (0-0)\,\mathbf{j} + (2x\cos(\pi y) - ax\pi\cos(\pi y))\,\mathbf{k} \end{aligned}$$

This to vanish for all $(x, y, z) \in \mathbb{R}^3$ requires $a = \frac{2}{\pi}$ and $b = -2$. As $\mathbb{R}^3$ is simply connected, F is conservative for $a = \frac{2}{\pi}$ and $b = -2$.

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$ be a potential function. Then

$$\frac{\partial f}{\partial x} = \frac{2}{\pi}x\sin(\pi y) \quad (1)$$

$$\frac{\partial f}{\partial y} = x^2\cos(\pi y) - 2ye^{-z} \quad (2)$$

$$\frac{\partial f}{\partial z} = y^2e^{-z} \quad (3)$$

Integrating (1) with respect to x gives

$$f(x, y, z) = \frac{1}{\pi}x^2\sin(\pi y) + g(y, z),$$

where $g(y, z)$ is an integration constant that can depend on y and z. Inserting this in (2) gives

$$x^2\cos(\pi y) + \frac{\partial g}{\partial y} = x^2\cos(\pi y) - 2ye^{-z}$$

which gives

$$\frac{\partial g}{\partial y} = -2ye^{-z}.$$

Integrating the latter equation with respect to y gives

$$g(y, z) = -y^2e^{-z} + h(z),$$

where $h(z)$ is an integration constant that can depend on z. So we have

$$f(x, y, z) = \frac{1}{\pi}x^2\sin(\pi y) - y^2e^{-z} + h(z).$$

Inserting this in (3) gives

$$y^2 e^{-z} + h'(z) = y^2 e^{-z}$$

which yields $h(z) = c$ for some constant $c \in \mathbb{R}$. So we get the potential function

$$f(x, y, z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + c.$$

(c) We have $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$. By the Fundamental Theorem of Line Integrals

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{j}) - f(\mathbf{i}) = -1 + c - c = -1.$$

4. We start by computing the left hand side of the equality. The curl of $\mathbf{F}$ is

$$\begin{aligned} \nabla \times \mathbf{F}(x, y, z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \\ &= -1\,\mathbf{i} - 1\,\mathbf{j} - 1\,\mathbf{k}. \end{aligned}$$

Parametrize the quarter disk S by

$$\phi(r, \theta) = (0, r\cos\theta, r\sin\theta)$$

with $(r, \theta) \in [0, 1] \times [0, \frac{\pi}{2}]$. From

$$\frac{\partial \phi}{\partial r}(r, \theta) = (0, \cos\theta, \sin\theta)$$

and

$$\frac{\partial \phi}{\partial \theta}(r, \theta) = (0, -r\sin\theta, r\cos\theta)$$

we get the normal vector

$$\frac{\partial \phi}{\partial r} \times \frac{\partial \phi}{\partial \theta} = (r\cos^2\theta + r\sin^2\theta)\,\mathbf{i} = r\,\mathbf{i}.$$

The normal vector is consistent with the given orientation on S. Hence

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_0^1 \int_0^{\pi/2} (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \phi}{\partial r} \times \frac{\partial \phi}{\partial \theta} \right) d\theta dr \\ &= \int_0^1 \int_0^{\pi/2} (-1, -1, -1) \cdot (r, 0, 0)\, d\theta dr \\ &= -\int_0^1 \int_0^{\pi/2} r\, d\theta dr \\ &= -\frac{\pi}{4}. \end{aligned}$$

We now compute the right hand side of the equality. The boundary of S consists of the three smooth pieces C_1 (the line segment between $(0,0,0)$ and $(0,0,1)$) C_2 (the line segment between $(0,0,0)$ and $(0,1,0)$) and C_3 (the arc in the (y,z)-plane from $(0,1,0)$ to $(0,0,1)$). These have parametrizations consistent with the orientation of S given by

$$\begin{aligned}\mathbf{r}_1(t) &= (0,0,(1-t)),\ t\in[0,1],\\ \mathbf{r}_2(t) &= (0,t,0),\ t\in[0,1],\\ \mathbf{r}_3(t) &= (0,\cos t,\sin t),\ t\in[0,\pi/2],\end{aligned}$$

respectively. Hence

$$\oint_{\partial S}\mathbf{F}\cdot \mathrm{d}\mathbf{s} = \int_{C_1}\mathbf{F}\cdot \mathrm{d}\mathbf{r}_1 + \int_{C_2}\mathbf{F}\cdot \mathrm{d}\mathbf{r}_2 + \int_{C_3}\mathbf{F}\cdot \mathrm{d}\mathbf{r}_3.$$

As $\mathbf{r}_1'(t) = (0,0,-1)$ is perpendicular to $\mathbf{F}(r_1) = (1-t)\,\mathbf{k}$ and $\mathbf{r}_2'(t) = (0,1,0)$ is perpendicular to $\mathbf{F}(r_2(t)) = t\,\mathbf{j}$, the first two integrals vanish. The third integral gives

$$\begin{aligned}\int_{C_3}\mathbf{F}\cdot \mathrm{d}\mathbf{r}_3 &= \int_0^{\pi/2}\mathbf{F}(r_3(t))\cdot\mathbf{r}_3'(t)\,\mathrm{d}t\\ &= \int_0^{\pi/2}(\cos t,\sin t,0)\cdot(0,-\sin t,\cos t)\,\mathrm{d}t\\ &= -\int_0^{\pi/2}\sin^2 t\,\mathrm{d}t\\ &= \frac{1}{2}(\sin t\cos t - t)\Big|_{t=0}^{t=\pi/2}\\ &= -\frac{\pi}{4}\end{aligned}$$

which agrees with the result above.